Baby Rudin 3rd Edition
Chapter 1: The Real and Complex Number Systems

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Highlights:

- **1.1 Example** - Show that $\sqrt{2}$ is not rational. Show that there are gaps in $\mathbb{Q}$. Use $q = p - \frac{p^2-2}{p+2} = \frac{2p+2}{p+2}$ and $q^2 - 2 = \frac{2(p^2-2)}{(p+2)^2}$.
- **1.9 Examples** - Show that $sup(E)$ and $inf(E)$ can either be part of E or not.
- **1.11 Theorem** - Show that an ordered set which has the *least upper bound property* also has the *greatest lower bound property*.
- **1.20 Theorem** - Show the Archimedean property of $\mathbb{R}$ and that $\mathbb{Q}$ is *dense* in $\mathbb{R}$.
- **1.21 Theorem** - Show that there is only one positive real y such that $y^n = x$.

Exercises:

1.1. If r is *rational* ($r \neq 0$) and x is *irrational*, prove that $r + x$ and rx are irrational.

Proof. If $r + x$ is rational, then for integers $m, n, q, p \in \mathbb{N}$, $x = m/n - p/q = (mq - np)/(nq)$, which is a rational number and a contradiction. If rx is rational, then for integers $m, n, q, p \in \mathbb{N}$, $x = (mq)/(np)$, which is a rational number and a contradiction. □

1.2. Prove that there is no rational number whose square is 12.

Proof. If $(m/n)^2 = 12$, then for integers m and n that do not have a common factor, $m^2/n^2 = 12 = 3 * 4 \implies m^2 = 3 * 4 * n^2$. 3 is a factor of m^2 and therefore m. If $m = (3r)$ with integer r, then $(3r)^2 = 3 * 4 * n^2 \implies 3r^2 = 4 * n^2$. 3 is also a factor of n^2 and therefore n, which is a contradiction. □

1.3. Prove proposition 1.15.

(a) If $x \neq 0$ and $xy = xz$ then $y = z$.

Proof. $xy = xz \implies (1/x)(xy) = (1/x)(xz) \implies y = z$

(b) If $x \neq 0$ and $xy = x$ then $y = 1$.

Proof. $xy = x \implies (1/x)(xy) = (1/x)x \implies y = 1$

(c) If $x \neq 0$ and $xy = 1$ then $y = 1/x$.

Proof. $xy = 1 \implies (1/x)(xy) = (1/x) \implies y = (1/x)$

(d) If $x \neq 0$ then $1/(1/x) = x$.

Proof. $1/(1/x) = ((x/1)(1))/((x/1)(1/x)) = x/1 = x$ □

1.4. Let E be a nonempty subset of an ordered set; suppose α is a lower bound on E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Proof. If $\alpha \leq x, \forall x \in E$ and $x \leq \beta, \forall x \in E$, then $\alpha \leq x \leq \beta \implies \alpha \leq \beta$ □

1.5. Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$inf(A) = -sup(-A).$$

Proof. If $\alpha = inf(A)$, then $\alpha \leq x, \forall x \in A$. If $\beta = -\alpha$, then $-\alpha \geq -x, \forall x \in A \implies x \leq \beta, \forall x \in -A$. Since α was the infimum of A, β is the supremum for $-A$. Therefore, $\alpha = inf(A)$ and $\beta = sup(-A)$ and because $-\alpha = \beta$, $inf(A) = -sup(-A)$. □

1.6. Fix $b > 1$.

(a) If m, n, p, q are integers, $n > 0, q > 0$, and $r = m/n = p/q$, prove that $(b^m)^{1/n} = (b^p)^{1/q}$. Hence it makes sense to define

$$b^r = (b^m)^{1/n}.$$

Proof. $b^r = (b^m)^{1/n} = b^{m/n} = b^{p/q} = (b^p)^{1/q}$ □

(b) Prove that $b^r + b^s = b^r b^s$ if r and s are rational.

Proof. If $r = m/n$ and $s = p/q$, then $b^{r+s} = b^{m/n+p/q} = b^{(mq+pn)/nq} = (b^{mq}b^{pn})^{1/nq} = b^{m/n}b^{p/q} = b^r b^s$ □

(c) If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \leq x$. Prove that

$$b^r = sup(B(r))$$

where r is rational. Hence it makes sense to define

$$b^x = sup(B(x))$$

for ever real x.

Proof. If $t \leq r$ and $b > 1$, then $b^t \leq b^r$ and b^r is an upper bound. If r is rational, then $t = r$ and $b^r \in B(r)$. If $b^r \in B(r)$, then $b^r \leq U$, for any upper bound U. Therefore $b^r = sup(B(r))$. □

(d) Prove that $b^{x+y} = b^x b^y$ for all real x and y.

Proof. From part *(c)*, $b^{x+y} = sup(B(x+y))$. If $b^t \in B(x+y)$, then $t \leq x+y$. If $t = r+s$ with $r \leq x$ and $s \leq y$, then $b^t = b^{r+s} = b^r b^s \leq sup(B(x))sup(B(y))$ and therefore $sup(B(x))sup(B(y))$ is an upper bound for $B(x+y)$. Now let's prove that it is also the least upper bound.

If $0 < U < sup(B(x))sup(B(y))$, then $\frac{U}{sup(B(x))} < sup(B(y))$. If $n = \frac{1}{2}[\frac{U}{sup(B(x))} + sup(B(y))]$, then $\frac{U}{sup(B(x))} < n < sup(B(y))$. $\exists k \in B(x)$ and $\exists v \in B(y)$ such that $\frac{U}{n} < k$ and $n < v \implies U = (\frac{U}{n}) * n < kv \in B(x+y)$. U is not an upper bound and therefore $sup(B(x))sup(B(y))$ is the least upper bound of $B(x+y)$. Therefore, $b^{x+y} = b^x b^y$. □

1.7. Fix $b > 1, y > 0$, and prove that there is a unique real x such that $b^x = y$, by completing the following outline. (This x is called the *logarithm of y to the base b.*)

(a) For any positive integer $n, b^n - 1 \geq n(b-1)$.

Proof. Induction base case ($n = 1$):

$b^1 - 1 \geq 1(b-1)$

Induction hypothesis ($n \geq 1$):

$b^n - 1 \geq n(b-1)$

For $n = 1$, $b^{n+1} \geq (n+1)(b-1) \implies bb^n - 1 \geq nb - n + b - 1 \implies b^{n+1} - b \geq n(b-1) \implies b(b^n - 1) \geq n(b-1)$, which holds due to induction hypothesis and $b > 1$. □

(b) Hence $b - 1 \geq n(b^{1/n} - 1)$.

Proof. Same as proof for *(a)* but with b replaced by $b^{1/n}$. □

(c) If $t > 1$ and $n > (b-1)(t-1)$, then $b^{1/n} < t$.

Proof. From part *(b)*, $b - 1 \geq n(b^{1/n} - 1) > (b-1)(t-1)(b^{1/n} - 1) \implies 1 > (t-1)(b^{1/n} - 1) \implies t > b^{1/n}(t-1) \implies t/(t-1) > b^{1/n}$. If $t > 1$, then $t > t/(t-1) > b^{1/n}$. □

(d) If w is such that $b^w < y$, then $b^{w+(1/n)} < y$ for sufficiently large n; to see this, apply part (c) with $t = yb^{-w}$.

Proof. From part *(c)*, $b^{1/n} < t \implies b^{1/n} < yb^{-w} \implies b^w(b^{1/n}) < yb^w(b^{-w}) \implies b^{w+1/n} < y$ □

(e) If $b^w > y$, then $b^{w-(1/n)} > y$ for sufficiently large n.

Proof. Same as proof for *(d)* but with $t = y^{-1}b^w$. □

(f) Let A be the set of all w such that $b^w < y$, and show that $x = sup(A)$ satisfies $b^x = y$.

Proof. If $x = sup(A)$, then there are three possible cases: $b^x < y$, $b^x > y$, or $b^x = y$. If $b^x < y$, then $x \in A$ and from part *(d)* $x + (1/n) \in A$ for sufficiently large n. This contradicts the fact that x is an upper bound for A. If $b^x > y$, then from part *(e)* $x - (1/n)$ for sufficiently large n. This contradicts the fact that x is the least upper bound. The only possibility left is that $b^x = y$. □

(g) Prove that this x is unique.

Proof. If s satisfies the above properties, then $b^s = y \implies b^s = y = b^x \implies b^s = b^x \implies 1 = b^x b^{-s} = b^{x-s} \implies x - s = 0 \implies x = s$. □

1.8. Prove that no order can be defined in the complex field that turns it into an ordered field. *Hint:* -1 is a square.

Proof. From definition 1.17 (ii) for an ordered field F: $xy > 0$ if $x \in F$, $y \in F$, $x > 0$, $y > 0$.

For the complex field, if $x = y = i$, then $xy = i^2 = -1 \not> 0$. □

1.9. Suppose $z = a + bi, w = c + di$. Define $z < w$ if $a < c$, and also if $a = c$ but $b < d$. Prove that this turns the set of all complex numbers into an ordered set. (This type of order relations is called a *dictionary order*, or *lexicographic order*, for obvious reasons.) Does this ordered set have the least-upper-bound property?

Proof. With the given constraints, z and w can have the cases:

$$a < c, b = d \to z < w$$

$$a < c, b < d \to z < w$$

$$a < c, b > d \to z < w$$

$$a > c, b = d \to z > w$$

$$a > c, b < d \to z > w$$

$$a > c, b > d \to z > w$$

$$a = c, b = d \to z = w$$

$$a = c, b < d \to z < w$$

$$a = c, b > d \to z > w$$

We know that the real numbers is an ordered set so that $a, b, c, d \in \mathbb{R}$ satisfy conditions (i) and (ii) of Definition 1.5. From the above cases we can see that z and w satisify (i). We will need to check that they meet (ii):

Let $t = e + fi$.

Cases ($a < c, a > c$): Let $z < w$ and $w < t$. Then we know that $a < c$ and $c < e$, where b, d, f can be any value. Since the real numbers are transitive, we see that $a < c, c < e \implies a < e$. Therefore, $z < w, w < t \implies z < t$. It is easy to see that the same reasoning applies for all the cases with $a > c$.

Cases ($a = c$): For the case that $a = c$ and $b = d$ we have the case of equality $z = w$. Obviously if $z = w$ and $w = t$ we have that $z = t$.

For the other cases in this group we can test them in tandem. For the case $a = c, b < d \implies z < w$ we can use z and w as is while for the case $a = c, b > d \implies z > w$ let us use t in place for z and leave w as is. Then we get that $a = c, b < d, e = c, f > d \implies a = c = e, b < d < f \implies a = e, b < f$. Therefore, we see that $z < w, w < t \implies z < t$.

This ordered set does not have the least-upper-bound property. To see this, let $B = \{(0, b) \mid b \in \mathbb{R}\}$ be a subset of our ordered set. This subset has an upper bound, since $(a, 0) > (0, b)$ for any $a > 0$. However, for any proposed least upper bound $(a, b), a > 0$, we see that

$$(a, b) > (\tfrac{a}{2}, b) > (0, b)$$

Where $(\frac{a}{2}, b)$ is an upper bound less than the proposed least upper bound, a contradiction. ◻

1.10. Suppose $z = a + bi, w = u + vi$, and

$$a = ((|w| + u)/2)^{1/2}, b = ((|w| - u)/2)^{1/2}.$$

Prove that $z^2 = w$ if $v \geq 0$ and that $\overline{z}^2 = w$ if $v \leq 0$. Conclude that every complex number (with one exception!) has two complex square roots.

Proof. $z^2 = (a + bi)(a + bi) = a^2 + 2abi - b^2 = \frac{|w|+u}{2} + 2(\frac{|w|+u}{2})^{1/2}(\frac{|w|-u}{2})^{1/2}i - \frac{|w|-u}{2} = u + ((|w| + u)(|w| - u))^{1/2}i = u + vi$. Note that here we used the fact that $(xy)^{1/2} = x^{1/2}y^{1/2}$. For $(\overline{z})^2$ we get the same equations except there is a negative sign for $-2abi$, which for $v \leq 0$ gives us the same answer. ◻

1.11. If z is a complex number, prove that there exists an $r \geq 0$ and a complex number w with $|w| = 1$ such that $z = rw$. Are w and r always uniquely determined by z?

Proof. Convert z to polar form. $z = a + bi \implies z = rcos(\theta) + rsin(\theta)i = r(cos(\theta) + sin(\theta)i) = rw$, where $w = cos(\theta) + sin(\theta)i$ and $|w| = (cos(\theta)^2 + sin(\theta)^2)^{1/2} = 1$. Yes, w and r are always uniquely determined by z because the phase and modulus depend on the complex number. ◻

1.12. If $z_1, \ldots, z_n$ are complex, prove that

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|.$$

Proof. Using induction with the same method that is used in Theorem 1.33 for proving part (e), which was the base case of n=2. $|z_1 + z_2 + \cdots + z_n| \leq |(z_1 + z_2 + \cdots + z_{n-1}) + z_n| \leq |z_1 + z_2 + \cdots + z_{n-1}| + |z_n| \leq |z_1| + |z_2| + \cdots + |z_{n-1}| + |z_n|$. □

1.13. If x, y are complex, prove that

$$||x| - |y|| \leq |x - y|.$$

Proof. This is the reverse triangle inequality.

$x = x - y + y \implies |x| = |(x - y) + y| \leq |x - y| + |y| \implies |x| - |y| \leq |x - y|$. Similar logic follows starting this equality for y, therefore $||x| - |y|| \leq |x - y|$. □

1.14. If z is a complex number such that $|z| = 1$, that is such that $z\overline{z} = 1$, compute

$$|1 + z|^2 + |1 - z|^2.$$

Proof. $|1+z|^2+|1-z|^2 = (1+z)\overline{(1-z)}+(1-z)\overline{(1-z)} = (1+z)(1+\overline{z})+(1-z)(1-\overline{z}) = 1+\overline{z}+z+1+1-\overline{z}-z+1 = 4$. [Note: $\overline{z + w} = \overline{z} + \overline{w}$ by Theorem 1.31 (a)] □

1.15. Under what conditions does equality hold in the Schwarz inequality?

The two sides of the Schwarz inequality are equal when the two vectors are linearly dependent.

1.16. Suppose $k \geq 3, x, y \in \mathbb{R}^k, |x - y| = d > 0$, and $r > 0$. Prove:

(a) If $2r > d$, there are infinitely many $z \in \mathbb{R}^k$ such that

$$|z - x| = |z - y| = r.$$

Proof. If $2r > d$ and $r = |z - x| = |z - y|$, then $2r = |x - z| + |z - y| > d = |x - y|$ [Theorem 1.37 (f)]. TODO - need to explicitly construct the k-2 dimensional sphere of solutions. □

(b) If $2r = d$, there is exactly one such z.

Proof. If $2r = d$, we can see from the equation in part (a) that there is exactly one such z. TODO - same as (a) □

(c) If $2r < d$, there is no such z.

How must these statements be modified if k is 2 or 1?

"a) you can construct explicitly the whole k - 2-dimensional sphere of solutions. (b) it is easy to see that there is one solution. Use some argument using the triangle inequality to rule out the existence of others. (c) just use the triangle inequality."

Proof. TODO

1.17. Prove that

$$|x + y|^2 + |x - y|^2 = 2|x|^2 + 2|y|^2$$

if $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^k$. Interpret this geometrically, as a statement about parallelograms.

Proof. If $x, y \in \mathbb{R}^k$, then by proof of Theorem 1.37 (e), $|x + y|^2 = (x + y) \cdot (x + y) = x \cdot x + 2x \cdot y + y \cdot y$ and $|x - y|^2 = x \cdot x - 2x \cdot y + y \cdot y$. Therefore, $|x + y|^2 + |x - y|^2 = 2x \cdot x + 2y \cdot y = 2|x|^2 + 2|y|^2$. □

1.18. If $k \geq 2$ and $x \in \mathbb{R}^k$, prove that there exists $y \in \mathbb{R}^k$ such that $y \neq 0$ but $xy = 0$. Is this also true if $k = 1$?

Proof. If $x = (x_1, x_2)$ and $y = (x_2, -x_1)$, then $x \cdot y = x_1 x_2 - x_2 x_1 = 0$. It is not true for $k = 1$ unless we are allowed to have $y = 0$. □

1.19. Suppose $a \in \mathbb{R}^k, b \in \mathbb{R}^k$. Find $c \in \mathbb{R}^k$ and $r > 0$ such that

$$|x - a| = 2|x - b|$$

if and only if $|x - c| = r$. (*Solution:* $3c = 4b - a, 3r = 2|b - a|$).

Proof.

1.20. With reference to the Appendix, suppose that property (III) were omitted from the definition of a cut. Keep the same definitions of order and addition. Show that the resulting ordered set has the least-upper-bound property, that addition satisfies axioms (A1) to (A4) (with a slightly different zero-element!) but that (A5) fails.

Proof. By getting rid of property (III) from the definition of a cut, the ordered set α now has a largest member. Thus, $\forall p \in \alpha$ we have $p \leq u$, where u is the largest member of α. u is an upper bound and additionally it is the least upper bound because $u \leq q$, $\forall q \in \mathbb{Q}$.

TODO - complete the other steps of the proof.